Equal Sign Conceptual Understanding and Early Algebra Problem Solving

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Abstract

Students often encounter difficulties when transitioning from arithmetic to algebra, potentially due to an incomplete understanding of mathematical symbols like the equal sign. We assessed students' equal sign conceptual understanding and algebra problem-solving strategies before and after algebra instruction. Participants engaged in a 40-minute session comprising various segments: an Equal Sign Conceptual Understanding Assessment, an Algebra Problem Solving Assessment, an intervention, a post-test Equal Sign Conceptual Understanding Assessment, and a post-test Early Algebra Problem Solving Assessment, followed by a Transfer Algebra Problem Solving Assessment. The intervention consisted of either a lesson on doing the same thing to both sides, a move-and-change strategy, or a control condition involving additional problem practice. Results showed students experienced challenges with both algebra equations and the conceptual understanding of the equal sign. Only 37% of participants correctly solved both algebra pretest problems, and 42% employed invalid strategies. Additionally, 43% of students lacked a firm grasp of the distinction between relations and operations, scoring below 5 points on the Equal Sign Conceptual Understanding Assessment. A significant correlation was found between students' conceptual understanding and problem-solving abilities at the pre-test stage. Students who used more advanced strategies for algebra problems displayed higher levels of conceptual understanding. Further research is needed to establish causal relationships between conceptual understanding and problem-solving, considering factors such as instructional impact and potential bidirectional influences between these skills.

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Equal Sign Conceptual Understanding and Early Algebra Problem Solving

Students often experience difficulty at the transition from arithmetic to algebra. One possible source for thus difficulty is students' incomplete conceptual understanding of some of the symbols used in mathematics, in particular, the equal sign. Previous studies have shown that younger students (preschoolers to fifth graders) have misconceptions about the equal sign's meaning. The present study examines students' understanding of the equal sign at the transition from arithmetic to algebra and explores the relationship between students' understanding of the equal sign and their problem solving skill (Perry, Church, & Goldin-Meadow, 1998; Rittle-Johnson & Alibali, in press).

Arithmetic and Algebra

The precise demarcation for the transition from arithmetic to algebra is debatable. Problems as uncomplicated as example 1 (Table 1.1) are sometimes considered algebra because it is not possible to state the problem without a variable. However, these problems can be solved directly with the definition of inverse operations. Children are taught within arithmetic that subtraction and division "undo" addition and multiplication respectively (Herscovics & Linchevski, 1994).

Table 1.1: Potential Problem Types for Early Algebra

#	EXAMPLE	DESCRIPTION
1	$y + 4 = 10$	equation with one operation to variable
2	$(z \times 2) + 4 = 10$	equation with multiple operations to variable
3	$(z \times 2) + 7 = 10 + z$	equation with multiple occurrences of the same variable

Fillow and Rojano (1984) sharply delineate algebra from arithmetic with a variable on both sides of a first degree equation (example 3). The presence of the variable on both sides of the equation necessitates directly operating on the variables. Operating on variables is a significant "cognitive gap" for students and may be a useful transition point for specifically studying variable understanding. Nevertheless, this may not be the most appropriate place to begin studying algebra generally. We do not know about difficulties students encounter while learning to repeatedly use inverse operations. Repeated use of inverse operations can not be done haphazardly or with only arithmetic definitions. Consequently in this study we focus on equations like example 2. Early algebra problems like example 2 cause students difficulty when written in symbolic form (Koedinger & Tabachneck, 1995).

Children's Difficulties with Algebra

Children have considerable difficulty solving early algebra problems such as "(x ~ 64) ÷ 3 = 26.5." This problem was solved correctly by only fifty percent of ninth grade students in a recent study (Koedinger & Tabachneck, 1995). Surprisingly, more students, seventy-five percent, could solve a corresponding algebra word problem. One possible explanation is the value of contextual information from word problems. However, Koedinger and Tabachneck (1995) also found students comparably successful with only a verbal description of the symbolic problem (see Table 1.2 for example). Students may be less successful with symbolic equations because of their greater familiarity with verbal representations. How are students using the verbal information differently from symbolic information?

Table 1.2: Representation of Early Algebra Problems (Koedinger & Tabachneck, 1995)

EXAMPLE	PROBLEM TYPE	CORRECTLY SOLVED BY 9^{TH} GRADERS
(x – 64) ÷ 3 = 26.5	symbolic	50%
After hearing that Mom won a lottery prize, Bill took the amount she won and subtracted the $64 that Mom kept for herself. Then he divided the remaining money among her 3 sons giving each $26.50. How much did mom win?	word/story	75%
Starting with some number, if I subtract 64 and then divide by 3, I get 26.5. What number did I start with?	verbal description	74%

Students are not translating verbal representations into symbolic representations. If they were, students would successfully solve symbolic problems at least as often as solving the other forms. An example of a typical student's written work (Table 1.3) shows they are solving the algebraic portion within the verbal presentation. Perhaps more surprising, incorrect work obtains the correct answer; 1063 + 217 does not equal 1280 – 425. What does students' performance suggest about their understanding of the equal sign? How might the equal sign be conceptually misunderstood to consider the typical students' solution sensible?

Table 1.3: Sample Algebra Word Problem with Typical 6th Grade Written Work

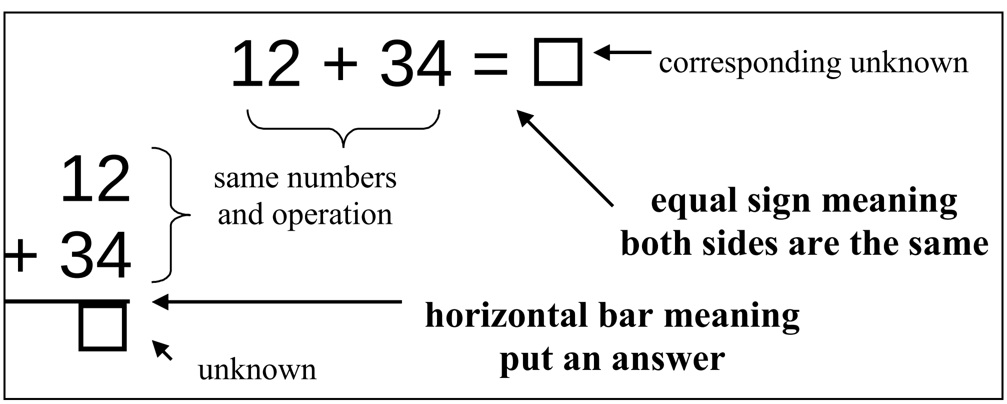
(Vergnanaud, Benhadj, & Dussouet, 1979 in Kieran, 1989)

WORD PROBLEM	In an existing forest, 425 new trees were planted. A few years later, the 217 oldest trees were cut. The forest then contained 1063 trees. How many trees were there before the new trees were planted?
TYPICAL STUDENT SOLUTION	1063 + 217 = 1280 ~ 425 = 855
SYMBOLIC ALGEBRA SOLUTION	(t + 425) ~ 217 = 1063 (t + 425) = 1063 + 217 (t + 425) = 1280 t = 1280 – 425 t = 855

Children's Equation Solving and Equal Sign Understanding

Comparing the symbolic algebraic solution and the typical students solution (table 1.3) shows how the correct answer is determined despite different equations. Both solutions perform the same operations in the same order. However, unlike the typical student solution, the algebraic solution preserves equality as the equation changes. The equal sign acts as a relation, a comparison of "the same as" between each equation side. The student's solution uses the equal sign to indicate putting an answer. The equal sign acts analogous to an operation, an indicator to "do something." Could adolescents beginning to learn algebra not understand the equal sign represents a relation?

Figure 1.1: An Arithmetic Problem in Two Typical Presentations



This possibility would not be far-fetched. Calculators use an equal sign key to compute an answer. Every part of a multiple digit arithmetic problem written in horizontal form has a corresponding part in vertical form except the equal sign (figure 1.1). When arithmetic is written horizontally it is usually presented as problem leading to an answer. For example, rarely do children see arithmetic problems like "☐ = 12 + 34." Most young children will not accept "☐ = 12 + 34" as a good problem because something must be done with the problem to equal an answer. They have an "operational", instead of "relational", understanding of the equal sign (Baroody and Ginsburg, 1983). Do these different conceptual understandings of the equal sign influence equation solving?

Previous research suggests students' conceptual understanding is related to their problem solving performance. Fourth grade students with a relational understanding of the equal sign are more likely to use valid strategies to solve equivalence problems (e.g. 3 + 4 + 5 = __ + 5) than students with an operational understanding (Rittle-Johnson & Alibali, in press). Further, instruction about a correct problem solving procedure was found to increase students' conceptual understanding of the equal sign. These findings highlight mutual causal influences between conceptual understanding problem solving procedures. In the present study, we examine whether there are similar relations between equal sign conceptual understanding and algebra problem solving strategies at the transition from arithmetic to algebra.

Hypotheses

The goal of this study is to investigate the relationships between equal sign conceptual understanding and early algebra equation solving. To do this, children's equal sign conceptual understanding and algebra problem solving strategies were assessed both before and after algebra instruction. Four specific hypotheses are examined.

First, we hypothesize that conceptual understanding of the equal sign and algebra problem solving strategies are related. That is, we expected that at the beginning of the study more sophisticated algebra strategies accompany more advanced equal sign conceptual understanding. If there is a mutual causal relationship between conceptual understanding and problem solving then, even in the absence of formal instruction, concepts and strategies should be related.

Second, we hypothesize that conceptual understanding would influence the effectiveness of learning strategies through instruction. Thus, we predicted that students with higher initial conceptual understanding would more often improve in problem solving strategies in response to instruction.

Third, we hypothesize that algebra instruction and advanced equal sign conceptual understanding enables problem solving on more advanced, novel, algebra problems. That is, transfer problems would be more successfully solved by those with sophisticated equal sign conceptual understanding and with instruction. Those students who already understand the equal sign's role have a stronger understanding of equation structure. Since algebra problem solving procedures rely on an equations structure, the procedures may be easier to generalize for those with higher conceptual understanding.

Fourth, we hypothesize that algebra instruction improves equal sign conceptual understanding. That is, participants in instructional conditions are more likely than those in a control condition to improve in conceptual understanding. If problem solving is a causal influence on conceptual understanding, then algebra instruction in a problem solving procedure should lead to improved equal sign conceptual understanding.

Method

Participants

Students not yet enrolled in Algebra from three Pittsburgh Pennsylvania private schools and one New Jersey public school participated. The 86 participants include 58 sixth grade and 28 seventh grade students. Three students were excluded from some analysis because the sessions were ended prior to completion. Students received a paper slide rule with instructions for returning the parental consent form. Those participating received pencils as a thank you gift.

Materials

A video camera recorded each session. Each session includes equation solving assessments conducted at a dry erase board and conceptual assessments conducted at a table.

Procedure

Each student participated in an individual forty-minute session that consisted of seven segments which are described in detail below: (a) pre-test with an Equal Sign Conceptual Understanding Assessment (b) the dog-cat-mouse game (Klahr, 1997) as a warm-up task for talk-aloud protocols (c) pre-test with an Early Algebra Problem Solving Assessment (d) intervention (e) post-test with an Equal Sign Conceptual Understanding Assessment (f) post-test with an Early Algebra Problem Solving Assessment (d) Transfer Algebra Problem Solving Assessment.

<u>Equal Sign Conceptual Assessment</u>

Five tasks were used to identify different aspects of a subject's equal sign understanding. Each task is scored independently with a 0, 1, or 2. Scores of 2 indicate the participant distinguishes among operations and relations and recognizes the equal sign as a relational symbol. Scores of 1 indicate the participant distinguishes among operations and relations but does not recognize the equal sign as a relational symbol. Scores of 0 indicate the participant does not distinguish among operations and relations; scores of 0 indicate viewing an equation as solely a string of symbols without an underlying structure. See table 2.2 for scoring details; if neither the 1 nor 2 score criteria is obtained, the score is 0. Please see appendix two for more detailed descriptions of each task.

encode

The encoding task identifies whether participants recognize the equal sign as a significant part of a math sentence. Participants were presented a math sentence for two seconds and then were asked to write down the problem. If at least three of four recollections are well formed math sentences, then the score is 2. A score of 1 indicates at least one well-formed math sentence.

symbol sort

The symbol sort task identifies which symbols, relations or operations, a participant considers the equal sign to most resemble. Participants sort seven symbols (2 numbers, 2 operations, & 3 relations including the equal sign) into piles "that belong together." Participants score 2 if they distinguish operations from relations and sort the equal sign with relations. A score of 1 indicates distinguishing operations from relations but sorting the equal sign with operations.

statement sort

The statement sort identifies recognition that the equal sign and other relations indicate a true or false relationship. If at least five of ten cards and at least two of three particularly difficult cards are sorted correctly, the score is 2. If one of these conditions is achieved the score is 1.

true sentence puzzle

The true sentence puzzle identifies the ability to construct true math sentences from the three defining properties of equality: reflexivity, symmetry, and transitivity. Puzzle pieces consist of the equal sign, the left equation side, and the right equation sign. The puzzle contains three equations using only small whole numbers. The right side of the first equation is the same number as the left side of the second equation. A third equation is a distracter. For example, the first two math sentences could be "1+2=3" and "3=9-6". The example "1+2=9-6" illustrates transitivity. Symmetry of equality means every quantity is equal to itself, "3=3". Reflexivity of equality means both sides of the equal sign are equal regardless of order, "3=1+2" demonstrates this. Completing at least three constructions and two of three construction types scores 2. If one of these conditions is achieved the score is 1.

define symbol

Participants indicate an explicit understanding of the equal sign by defining the symbol. After a first answer participants are asked "can it mean anything else?" If the first definition indicates a relational understanding, the score is 2. If the second definition indicates a relational understanding, the score is 1.

Table 2.2: scoring of equal sign conceptual assessment

TASK	SCORE OF 1	SCORE OF 2
encode	one well-formed math sentence	three well formed math sentences
symbol sort	distinguishing operations from relations but sorting the equal sign with operations	distinguish operations from relations and sort the equal sign with relations
statement sort	five target cards or two of three particularly difficult cards	five target cards and two of three particularly difficult cards
sentence puzzle	three constructions or two of three construction types	three constructions and two of three construction types
define symbol	second definition is relational	first definition is relational

<u>Early Algebra Equation Solving Assessment</u>

Early algebra problems with two-inch high symbols were presented on erasable sheets. Each assessment consists of two math sentences each with one unknown. All math sentences were of the form "($\square \times 9) - 63 = 45$". Please see appendix three for the rules generating math sentences and actual problems chosen for the assessments. Each algebra problem in an assessment was scored for correctness and strategy. Correctness score is either "true" for the variable's values which makes the math sentence true, "false" for any other number, or "give up" for non-numerical answers. Strategy score is either "domain specific" for strategies specific to algebra (exp.: isolate the variable and do the same thing to both sides), "domain general" for strategies widely applicable to problem solving (exp.: generate & test and set sub-goal), and "invalid" for strategies without a mathematical justification (exp.: guess and make new problem from problem parts). See table 2.1 for examples of strategy use.

Table 2.1: algebra equation strategies

SCORE	STRATEGY	PIECE OF A PROTOCOL WITH (☐ × 9) – 63 = 45
Invalid	Make new Problem	It's 7 because 7 × 9 = 63
Invalid	Combine Parts	It's 54 since 63 - 9 is 54
General	Generate & Test	10 × 9 is 90 and 90 – 63 is 27, so it's bigger than 10.
General	Sub-Goal	What minus 63 is 45?
Specific	Move & Change	First add 63 to 45.
Specific	Balance	First add 63 to both sides.

Transfer Algebra Equation Solving Assessment

The three transfer problems were presented like the early algebra problems. Numbers and operations were chosen in the same manner. Each transfer problem is a single alteration of the early algebra problems' equation structure: either terms combined by an operation are exchanged, "7 × (☐ – 21) = 49" and "(84 ÷ ☐) + 9 = 16", or there is an additional step, "((☐ × 2) – 12) ÷ 4 = 8". The scoring of this assessment is identical to the Early Algebra Equation Solving Assessment.

Interventions

Each students participated individually in one of the three randomly assigned interventions: two algebra lessons or a control condition. The two lessons were based on the traditional ways algebra procedures are taught in schools (Pirie and Martin, 1997). Please see appendix one for the complete algebra lesson dialogues.

balance strategy instructional condition.

The balance instruction emphasizes "whatever we do to one side we do to the other" with an example. Students learn a procedure using inverse operations to get an answer in the form "□ = #." Neither the equal sign nor the term "operation" is explicitly referenced. Following instruction, participants are guided through solving a second example.

move and change strategy instructional condition.

The move and change instruction emphasizes "move each number and change the operation to isolate the variable" with an example. Students learn a procedure using inverse operations to get an answer in the form "□ = #." Neither the equal sign nor the term "operation" is explicitly referenced. Following instruction, participants are guided through solving a second example.

control condition.

The control condition attempted to solve two early algebra problems. The first problem is the same example other students saw during instruction. The second problem is the same example other students were guided through.

Reliability

Reliability was assessed with 16 random participants. An independent rater coded both pre-tests to estimate reliability of conceptual understanding and problem solving assessments. The rater coded the sessions from video tape, and for the encoding task, the students' written responses were coded. The correlation between raters for pre-test equal sign conceptual understanding scores was .93. Agreement of problem solving codes for the problem solving strategies was 78%.

Results

The results address four aspects of the relationships between conceptual understanding of the equal sign and algebra problem solving: (1) the relationship between conceptual understanding and problem solving prior to instruction (2) change in problem solving due to instruction and conceptual understanding (3) instructional and conceptual influences on solving of novel algebra problems, and (4) change in conceptual understanding from instruction.

Initial Conceptual Understanding and Initial Problem Solving

We hypothesized that more advanced conceptual understanding and more advanced problem solving are related. Specifically we predicted that prior to the instructional intervention, students using more sophisticated algebra problem strategies have higher equal sign conceptual understanding scores. As predicted, conceptual understanding scores are greatest for those students who used a domain-specific strategy, and lowest for those students who used invalid strategies ($F(2,83) = 9.01$, $p < .001$; See Table 3.1). Post-hoc analyses show that students who used domain-specific strategies had significantly higher scores than student who used both invalid strategies and those who used domain-general strategies. However, scores of students who used invalid strategies and scores of students who used domain-general strategies did not differ significantly..

Table 3.1: relationship of initial conceptual understanding and problem solving

BEST PRE-TEST ALGEBRA STRATEGY	N	AVG CONCEPTUAL UNDERSTANDING
invalid	10	2.90
domain general	30	4.13
domain specific	46	5.54

Change in Problem Solving and Instruction

We expected instruction to improve problem solving since the interventions are modeled after typical classroom curriculum. The proportion of students whose strategies improved between the two problem solving assessments are expected to be greater for the instructional groups than the control. Students who used domain-specific strategies on both pre-test problems were excluded from the analysis since they could not improve. In addition, participants whose strategy change could not be clearly assessed were also excluded (e.g., change from a domain-specific and an invalid strategy on pretest to two domain-general strategies on posttest). As predicted, more students improved their problem solving in each of the instructional groups than in the control (see Table 3.2). Students learning the move-and-change strategy improved their problem solving more than the control (Wald $X^2 = 5.59$, $p < .02$). Students learning the balance strategy also improved their problem solving more than the control; however the effect was only marginally significant (Wald $X^2 = 3.49$, $p = .061$).

Table 3.2: change in problem solving due to instruction

INSTRUCTION	N	% WHO IMPROVE IN PROBLEM SOLVING
balance	20	70
move & change	24	71
control	22	36

As seen in table 3.3, the instructions show a similar prior conceptual understanding pattern for the effectiveness of instruction. Students with higher equal sign conceptual understanding (CU scores of 5 to 10) were more likely to learn from instruction than those with lower (scores of 0 to 4) conceptual understanding. The control group did not show this pattern.

Table 3.3: change in problem solving due to instruction, by pre conceptual understanding

INSTRUCTION	INITIAL CONCEPTUAL UNDERTSANDING	N	% WHO IMPROVE IN PROBLEM SOLVING
balance	low	7	57
balance	high	13	77
move & change	low	14	57
move & change	high	10	90
control	low	9	44
control	high	13	31

Transfer of Problem Solving and Instruction

Instruction in early algebra problems might be expected to transfer to other similar algebra problems. After completing the instruction and repeating the problem solving and conceptual understanding assessments, students attempted three transfer problems. We predict that students given instruction would be more likely to answer a problem correctly using with a domain specific strategy. Students already using domain specific strategies at the problem solving pre-test were excluded from analysis. This removes those students whose problem solving strategies may not have been able to benefit from the instruction. As predicted, more students given each instruction correctly solved at least one transfer problem with a domain specific strategy (See Table 3.4).

Table 3.4: transfer of domain specific strategy due to instruction

INSTRUCTION	N	% WHO CORRECTLY USE A DOMAIN SPECIFIC STRATEGY
balance	21	33
move & change	22	50
control	22	14

Those who learned move-and-change transferred knowledge more often than the control condition (Wald $X^2 = 6.00$, $p < .05$). Although students learning the balance strategy improved more often than the control, the effect did not reach significance (Wald $X^2 = 0.90$, $p = .34$). As seen in Table 3.5, the effectiveness of balance instruction for transfer problems appears to be influenced by initial equal sign conceptual understanding. Students with high conceptual understanding in the balance condition learn the balance strategy transfer at a rate similar to high sonceptual understanding students in the move-and-change instruction. Balance instruction students with low conceptual understanding are similar to low conceptual understanding students in the control.

Table 3.5: transfer of domain specific strategy due to instruction

INITIAL CONCEPTUAL UNDERTSANDING	INSTRUCTION	N	% CORRECTLY USE DOMAIN SPECIFIC STRATEGY
low	balance	8	13
low	move & change	13	46
low	control	9	11
high	balance	13	46
high	move & change	9	56
high	control	13	15

Conceptual Understanding due to Instruction

We predicted that instruction about problem solving will improve conceptual understanding since algebra problem solving relies on an equation's structure. The proportion of students whose conceptual understanding improved from pre to post equal sign assessments is expected to be greater for the instructional groups than the control. Those with conceptual understanding pre-test scores of 9 and 10 are excluded due to insufficient room for potential improvement. As seen in table 3.6, students learning the move-and-change strategy improved their conceptual understanding more than the control (Wald $X^2 = 4.28$, $p < .05$) but students learning the balance strategy did not.

Table 3.6: change in conceptual understanding due to instruction

INSTRUCTIONAL CONDITION	N	% IMPROVE CONCEPTUAL UNDERSTANDING
Balance	28	54
Move & Change	27	74
Control	25	44

One possible explanation for the instructions different effect on conceptual understanding is the conceptual prerequisites. As seen in table 3.7, move-and-change instruction promotes conceptual understanding more than the control for both low and high initial equal sign conceptual understanding. However, balance instruction only promotes equal sign conceptual understanding beyond the control if initial conceptual understanding is high (See table 3.7)

Table 3.7: change in conceptual understanding due to instruction, by conceptual groups

CONDITION	INITIAL CONCEPTUAL UNDERTSANDING	N	% IMPROVE CONCEPTUAL UNDERSTANDING
balance	low	10	50
balance	high	18	56
move & change	low	15	87
move & change	high	12	58
control	low	12	67
control	high	13	23

Discussion

The present study shows some qualities of sixth and seventh grade students' conceptual understanding and problem solving. The students participating come from neighborhoods with widely varying socio-economic status and attend different types of schools. Though this was not a perfectly representative sample, the results are suggestive of the difficulties experienced by a wide range of students. Clearly, sixth and seventh grade students have difficulties with algebra equations. Only 37% of our participants solved both algebra pretest problems correctly and 42% of our participants used an invalid strategy on at least one of those problems. Further, sixth and seventh grade students also do not fully understand equations or the equal sign. Forty-three percent (43%) of students did not demonstrate firm knowledge of the difference between relations and operations; they scored under 5 of the 10 points on the pre-test of equal sign conceptual understanding.

We also found a relationship at pre-test between students' conceptual understanding and problem solving. Those students using more advanced strategies for algebra problems averaged higher levels of conceptual understanding of the equal sign. This may be surprising since the two assessments are very different. Aside from some of the encoding problems, no algebra appears in any of the equal sign conceptual understanding tasks. The algebra instruction neither contained references to the equal sign nor used the terms "operation" and "relation." There are several potential explanations for the association, any number of which may be true: (1) Students may use their conceptual understanding to construct strategies; (2) the use of successful strategies may promote conceptual understanding; or (3) a third source, such as instruction, may promote both conceptual understanding and problem solving. Further research is needed to determine the precise causal relationships.

We also found instructional effects both on problem solving and on conceptual understanding. It is not surprising instruction led to improved problem solving from pre-test to post-test. The lessons were designed to teach an algebraic procedure for those specific problems. Some students successfully generalized their learned procedure to transfer problems. However, this was only statistically significant for the move-and-change instruction. It may be more surprising, since the instruction focused on a procedure, that students in the move-and-change instructional group improved conceptual understanding from pre-test to post-test. That is, the instruction did not explicitly teach concepts. This conceptual change did not occur for those students in the balance instructional group. Thus, the two instructional groups differed in both problem solving and conceptual understanding.

No statistically significant explanation of the differences between instructional lessons was found. However, a consistent suggestive pattern emerges. When we examine only students with low initial equal sign conceptual understanding, balance instruction is typically comparable to the control condition. However, when we examine only students with high initial equal sign conceptual understanding, balance instruction is typically comparable to the move-and-change instruction. Thus, there may be different conceptual prerequisites for the two types of instruction. This suggests instruction design may benefit from examining the conceptual prerequisites in addition to procedural prerequisites of a problem-solving lesson. For example, prior to teaching algebra, a teacher might first provide conceptual instruction on the structure of an equations. This may be effective since previous research indicates instruction on conceptual understanding of the equal sign is effective even for young children (Baroody and Ginsburg, 1983).

In this study, we investigated some of the many possible relationships among conceptual understanding, problem solving, and instruction. Further research will be needed to specify these relations more precisely. However, at a minimum, the present study provides suggestive evidence that may help address difficulties in early algebra instruction. The findings demonstrate that middle school students do not fully understand the meaning of the equal sign symbol. This incomplete understanding may be one source of students' difficulty at the transition from arithmetic to algebra.

Algebra Lessons

Script for Move-and-Change Instruction

Introduction to Instruction

- ✓ Do you want to see how mathematicians solve problems like these? ...
- ✓ When we have a problem like this:

🗁 *put problem on board*: (☐ + 6) ÷ 8 = 7

- ✓ We want to try and get something like this, where the square is equal to a number:

✍ ☐ = ?

- ✓ To get from the orginal math sentence to the number for the square we'll work backwards. First look at what arithmetic tells us to do with the square.

Instruction with an Example

- ✓ In our example first we first add six to the box because parentheses say "do me first", then we divide that by eight. *gesture to problem on board*. So the eight is where we'll start.
- ✓ To solve this move and change parts of the problem from one side to the other so we get the box alone.
- ✓ Use the same numbers, but but do a different thing. *write operation pairs on board.*

- ✓ when you see "+" move it to the other side and "-"
- ✓ when you see "-" move it to the other side and "+"
- ✓ when you see "×" move it to the other side and "÷"
- ✓ when you see "÷" move it to the other side and "×"

✓ Look at this example: *gesture to problem attached to board* (☐ + 6) ÷ 8 = 7

✓ first we'll start with the number 8 and we'll move it to the other side and multply since now there's division:

✍ (☐ + 6) = 7 × 8

✓ the rest of this side stays the same and 7 × 8 is 56. *gesture.*

✍ (☐ + 6) = 56

✓ we can drop parenthese if they're all the way on the outside:

✍ ☐ + 6 = 56

✓ use the "6" but this time move and "-":

✍ ☐ = 56 – 6

✓ the square is now left alone and 56 - 6 is 50. *gesture.*

✍ ☐ = 50

✓ and that's right because this math sentence is true! *go through*

✍ (50 + 6) ÷ 8 = 7

<u>A Walked Through Example</u>

✓ Now let's try one together!

🗁 *put problem on board*: (☐ ÷ 3) – 5 = 9

✓ what do the parentheses say is the first thing that happens to the box? *(if wrong or no answer) ÷3*

✓ what happens after that? *– 5*

✓ then what number should we start with? *5*

✓ since it's now subtracting 5, what should we do to move things from one side to the other and get the box closer to alone? *add 5 to the other side*

✍ (☐ ÷ 3) = 9 + 5

✓ what is 9 + 5? *14*

✓ what's left after we move the 5 from this side? *(☐ ÷ 3)*

✍ (☐ ÷ 3) = 14

✓ what can we do with parentheses when they're all the way on the outside? *drop them*

✍ ☐ ÷ 3 = 14

✓ since the problem is dividing the square by 3, what should we move and what should we change to get the box alone? *move and multiply by 3*

✍ ☐ = 14 × 3

✓ what is 14 × 3? it's okay if we do some work on the side. *42*

✓ what's left after we move the 3 on this side? *☐*

✍ ☐ = 42

✓ and that's right! *(write number in square)* because 42÷ 3 is 14, 14 minus 5 is equal to 9

✓ ready to try some on your own?

Script for Balance Instruction

Introduction to Instruction

- ✓ Do you want to see how mathematicians solve problems like these? ...
- ✓ When we have a problem like this:

🗁 *put problem on board*: (☐ + 6) ÷ 8 = 7

- ✓ We want to try and get something like this, where the square is equal to a number:

✍ ☐ = ?

- ✓ To get from the orginal math sentence to the number for the square we'll work backwards. First look at what arithmetic tells us to do with the square.

Instruction with an Example

- ✓ In our example first we first add six to the box because parentheses say "do me first", then we divide that by eight. *gesture to problem on board*. So the eight is where we'll start.
- ✓ To solve this make the same changes to both sides in the same way so the math sentence stays balanced.
- ✓ Use the same numbers, but but do a different thing. *write operation pairs on board.*
    - ✓ when you see "+" then "-" from both sides
    - ✓ when you see "-" then "+" it to both sides
    - ✓ when you see "×" then "÷" to both sides
    - ✓ when you see "÷" then "×" both sides

✓ Look at this example: *gesture to problem attached to board* (☐ + 6) ÷ 8 = 7

✓ first we'll start with the number 8 and we'll multply it to both sides since now there's division:

✍ (☐ + 6) ÷ 8 × 8 = 7 × 8

✓ we have something that's ÷8 and then ×8 so it's still the same thing. *gesture.*

✍ (☐ + 6) = 56

✓ we can drop parenthese if they're all the way on the outside:

✍ ☐ + 6 = 56

✓ use the "6" but this time "–" both sides:

✍ ☐ + 6 – 6 = 56 – 6

✓ if you've got anything in the square and you first +6 and then -6 you'll still have the same thing in the square. *gesture.*

✍ ☐ = 50

✓ and that's right because this math sentence is true! *go through*

✍ (50 + 6) ÷ 8 = 7

A Walked Through Example

✓ Now let's try one together!

🗁 *put problem on board*: (☐ ÷ 3) – 5 = 9

✓ what do the parentheses say is the first thing that happens to the box? *(if wrong or no answer) ÷3*

✓ what happens after that? *– 5*

✓ then what number should we start with? *5*

✓ since it's now subtracting 5, what should we do to both sides so the problems stays balanced? *add 5 to both sides*

✍ (☐ ÷ 3) – 5 + 5 = 9 + 5

✓ what is 9 + 5? *14*

✓ what's left after we cancel out subtracting 5 and then adding 5? *(☐ ÷ 3)*

✍ (☐ ÷ 3) = 14

✓ what can we do with parentheses when they're all the way on the outside? *drop them*

✍ ☐ ÷ 3 = 14

✓ since the problem is dividing the square by 3, what should we do to both sides so the problems stays balanced? *multiply both sides by 3*

✍ ☐ ÷ 3 × 3 = 14 × 3

✓ what is 14 × 3? it's okay if we do some work on the side. *42*

✓ what's left after we divide by 3 and multiply by 3? *☐*

✍ ☐ = 42

✓ and that's right; it makes our original math sentence true! *(write number in square)* because 42÷ 3 is 14, 14 minus 5 is equal to 9

✓ ready to try some on your own?

Equal Sign Conceptual Understanding Assessment

Script for Conceptual Understanding Assessment

encode

✓ I'm going to show you a math problem for a short time. Then I'll put it down and I'd like you to write it on this card exactly as you saw it. Don't try to solve it. Just write it as close to what you saw as you can.

🗁 *show four problems one at a time. hold card up for a count of three, put problem down. subjects should write each problem on a separate card.*

symbol sort

✓ I'm going to give you a pile of card with symbols on them. They're all things used in math, but you probably don't use them all the same way. I'd like you to separate them into three piles of symbols that belong together.

🗁 *place template on table*

🗁 *hand subject pile of cards with symbols*

☺ Good job.

<u>statement sort</u>

✓ Now we'll do something similar except this time there will be four piles. I've already labeled the piles. We'll go through a few examples so you can see what they mean.

🗁 *place template on table*

🗁 What do think of this? *Show card at top of stack* Yeah, that's silly. It's not true; this is false so it goes in the "math that's false" pile. *Place card beside "math that's false" pile.*

🗁 What about this one? *Show card at top of stack* Right, *read card*. It's true so it goes in the "math that's true" pile. *Place card beside "math that's true" pile.*

🗁 What would you think when you see this? *Show card at top of stack.* That's right! It's#. It's not really true or false but it's real math. It goes in the "math that's not true or false" pile. *Place card beside "math that's not true" or false pile.*

🗁 What about this one? *Show card at top of stack.* I'm puzzled too. *name symbols on card* are all from math. But all together it's not true or false or a problem. It's just nonsense that looks kind of like math. So this goes in the "looks like math but it's not" pile. *Place card beside "looks like math but it's not" pile.*

🗁 *Give subject remaining cards.*

☺ Great!

<u>true sentence puzzle</u>

🗁 These are pieces of a puzzle. *Spread puzzle pieces out on table.* But instead of fitting together with those little grooves, they fit together to make true math sentences. The neat thing with this puzzle is you can put parts together lots of ways. One way I thought of is to *assemble three sentences: first the conventional order, atypical order, and then distractor.*

- ✓ Can you take any three pieces and put them together a new way to make a true math sentence? It's okay if your sentence means something like one of mine, just put the parts together differently.
- ✓ Can you think of another way to put three pieces together? *Repeat until subject says "no"*
- ☺ Good!

<u>define symbol</u>

- ✓ One of the math symbols we use is the equal sign. *Hold up card from puzzle task with equal sign.* What does the equal sign mean?
- ✓ Can it mean anything else?

Materials for Conceptual Understanding Assessment

<u>encode</u>

version a

- early algebra: $(\square - 6) \div 3 = 1$
- arithmetic: $(7 + 6) - 8 = \square$
- arithmetic and one step 'problem' side: $(2 - 3) + 1 = 9 - \square$
- arithmetic and one step 'answer' side: $(1 + 5) \times 8 \div 2 = \square$

version b

- early algebra: $(\square \div 3) + 5 = 8$
- arithmetic: $(7 - 6) \times 9 = \square$
- arithmetic and one step 'problem' side: $(5 - 1) \times 2 = 4 + \square$
- arithmetic and one step 'answer' side: $(8 \div 2) + 9 - 7 = \square$

<u>symbol sort</u>

version a

- symbol cards: $1 \quad + \quad \div \quad < \quad > \quad 8 \quad =$

version b

- symbol cards: $2 \quad \times \quad + \quad > \quad < \quad 5 \quad =$

<u>statement sort</u>

version a

statement cards	8 > 12	11 > 7	10 × 11	$ 12 (	12 + 1
5 < 10	11 < 7	5 % 5	11 = 9	10 = 10	12 ÷ 6
÷1−7÷	6×9÷3	8+1 = 9	8÷4 = 5	5 > 9−2	3 = 12÷4
3×1÷3−1	7−2 = 7−2	6+4 < 6+4	10−1×2+3	3+4 > 6×7	11−1 = 2×5

version b

statement cards	6 > 11	12 > 7	11 + 10	) 4 %	10 ÷ 5
9 < 12	10 < 8	3 $ 6	9 = 10	11 = 11	12 − 2
÷5+8+	11−1×3	2×6 = 12	7+11 = 10	1 > 12÷6	3 = 8−5
11+1×11−1	3+7 = 3+7	2×5 > 2×5	11+1×9−6	12÷3 < 7−4	4+1 = 10÷2

<u>true sentence puzzle</u>

version a

🗁 template showing three solutions of the puzzle: 1+2 = 3 3 = 27÷9 2×6 = 12

version b

🗁 template showing four solutions of the puzzle: 2×4 = 8 8 = 15-7 4+8 = 12

<u>define symbol</u>

examples of relational equal sign responses

both sides are the same

one number is the same as the other number

same as

same exact thing

same amount of something

same thing, same problem

that something is the same value

the numbers on both sides of it have the same value

the same, like if you were to compare something

the stuff that's over here is the same as this

they're equal on both sides, the same number

when comparing things they'd have to be the same thing

examples of operational equal sign responses

all the problem together is the answer afterwards

answer to a problem

it comes before an answer

if you have a plus or any type of problem it equals to the answer

shows what the answer is

shows what you get when you add, subtract, multiply, or divide

solve the problem, there's the answer

tells you what it's going to be like

there's the answer

what the total is

what you put together is the total

Early Algebra Problem Solving Assessment

Squares always represent variables to minimize misconceptions. Children are familiar with a "□" as an unknown. Seventh through ninth grade students, including those instructed in algebra, frequently have misconceptions about the role of letters as variables (MacGregor and Stacey, 1997). Two operations with whole numbers upon the unknown may yield a single whole number on the right side of the equal sign. The left-most operation is always in parentheses with the unknown. The second operation is always chosen so the problem could not be simplified with just the left equation side. The unknown is always a whole number and all steps in a valid algebraic strategy only involve whole numbers. Incorrectly ordered algebraic steps also only involve whole numbers, in the example $45 \div 9 = 5$ and $5 + 63 = 68$.

problems for version one

- 🗁 an early algebra problem: $(\square \times 9) - 63 = 45$
- 🗁 an early algebra problem: $(\square + 18) \times 3 = 72$

problems for version two

- 🗁 an early algebra problem: $(\square \times 3) + 12 = 51$
- 🗁 an early algebra problem: $(\square - 49) \times 7 = 14$

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